Problem 4.8

Starting from the Rodrigues formula, derive the orthonormality condition for Legendre polynomials:

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \left(\frac{2}{2\ell+1}\right) \delta_{\ell\ell'}. \tag{4.34}$$

Hint: Use integration by parts.

Solution

Suppose that $\ell \neq \ell'$ for now. $P_\ell(x)$ and $P_{\ell'}(x)$ are Legendre polynomials, so they both satisfy Legendre's differential equation (Equation 2.208 on page 90).

$$\begin{cases} (1-x^2)\dfrac{d^2 P_\ell}{dx^2} - 2x\dfrac{dP_\ell}{dx} + \ell(\ell+1)P_\ell(x) = 0 \\ (1-x^2)\dfrac{d^2 P_{\ell'}}{dx^2} - 2x\dfrac{dP_{\ell'}}{dx} + \ell'(\ell'+1)P_{\ell'}(x) = 0 \end{cases}$$

$$\begin{cases} \dfrac{d}{dx}\left[(1-x^2)\dfrac{dP_\ell}{dx}\right] + \ell(\ell+1)P_\ell(x) = 0 \\ \dfrac{d}{dx}\left[(1-x^2)\dfrac{dP_{\ell'}}{dx}\right] + \ell'(\ell'+1)P_{\ell'}(x) = 0 \end{cases}$$

Multiply both sides of the first equation by $P_{\ell'}(x)$, and multiply both sides of the second equation by $P_\ell(x)$.

$$\begin{cases} \dfrac{d}{dx}\left[(1-x^2)\dfrac{dP_\ell}{dx}\right] P_{\ell'}(x) + \ell(\ell+1)P_\ell(x)P_{\ell'}(x) = 0 \\ \dfrac{d}{dx}\left[(1-x^2)\dfrac{dP_{\ell'}}{dx}\right] P_\ell(x) + \ell'(\ell'+1)P_\ell(x)P_{\ell'}(x) = 0 \end{cases}$$

$$\begin{cases} \ell(\ell+1)P_\ell(x)P_{\ell'}(x) = -\dfrac{d}{dx}\left[(1-x^2)\dfrac{dP_\ell}{dx}\right] P_{\ell'}(x) \\ -\ell'(\ell'+1)P_\ell(x)P_{\ell'}(x) = \dfrac{d}{dx}\left[(1-x^2)\dfrac{dP_{\ell'}}{dx}\right] P_\ell(x) \end{cases}$$

Add the respective sides of these equations.

$$[\ell(\ell+1) - \ell'(\ell'+1)]P_\ell(x)P_{\ell'}(x) = \frac{d}{dx}\left[(1-x^2)\frac{dP_{\ell'}}{dx}\right] P_\ell(x) - \frac{d}{dx}\left[(1-x^2)\frac{dP_\ell}{dx}\right] P_{\ell'}(x)$$

Integrate both sides with respect to x from -1 to 1.

$$\int_{-1}^{1}[\ell(\ell+1)-\ell'(\ell'+1)]P_\ell(x)P_{\ell'}(x)\,dx=\int_{-1}^{1}\left\{\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_{\ell'}}{dx}\right]P_\ell(x)-\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_\ell}{dx}\right]P_{\ell'}(x)\right\}dx$$

Bring the constants in front of the integral on the left, and split up the integral on the right. Integrate the resulting integrals by parts.

$$\begin{aligned}
[\ell(\ell+1)-\ell'(\ell'+1)]\int_{-1}^{1}P_\ell(x)P_{\ell'}(x)\,dx &= \int_{-1}^{1}\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_{\ell'}}{dx}\right]P_\ell(x)\,dx \\
&\quad -\int_{-1}^{1}\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_\ell}{dx}\right]P_{\ell'}(x)\,dx \\
&= \left[\overbrace{\left(1-x^2\right)}^{=0}\frac{dP_{\ell'}}{dx}\right]P_\ell(x)\Bigg|_{-1}^{1}-\int_{-1}^{1}\left(1-x^2\right)\frac{dP_{\ell'}}{dx}\frac{dP_\ell}{dx}\,dx \\
&\quad -\left[\underbrace{\left(1-x^2\right)}_{=0}\frac{dP_\ell}{dx}\right]P_{\ell'}(x)\Bigg|_{-1}^{1}+\int_{-1}^{1}\left(1-x^2\right)\frac{dP_\ell}{dx}\frac{dP_{\ell'}}{dx}\,dx \\
&= \int_{-1}^{1}\left(1-x^2\right)\frac{dP_\ell}{dx}\frac{dP_{\ell'}}{dx}\,dx-\int_{-1}^{1}\left(1-x^2\right)\frac{dP_{\ell'}}{dx}\frac{dP_\ell}{dx}\,dx \\
&= \int_{-1}^{1}\left(1-x^2\right)\left(\frac{dP_\ell}{dx}\frac{dP_{\ell'}}{dx}-\frac{dP_{\ell'}}{dx}\frac{dP_\ell}{dx}\right)dx \\
&= \int_{-1}^{1}\left(1-x^2\right)(0)\,dx \\
&= 0
\end{aligned}$$

Therefore, if $\ell\neq\ell'$,

$$\int_{-1}^{1}P_\ell(x)P_{\ell'}(x)\,dx=0.$$

Suppose now that $\ell = \ell'$. Then

$$\begin{aligned}
\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\,dx &= \int_{-1}^{1} P_\ell(x)P_\ell(x)\,dx \\
&= \int_{-1}^{1} \left[\frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}\left(x^2-1\right)^\ell\right]\left[\frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}\left(x^2-1\right)^\ell\right] dx \\
&= \frac{1}{(2^\ell)^2(\ell!)^2}\int_{-1}^{1} \left[\frac{d^\ell}{dx^\ell}\left(x^2-1\right)^\ell\right]\left[\frac{d^\ell}{dx^\ell}\left(x^2-1\right)^\ell\right] dx \\
&= \frac{1}{2^{2\ell}(\ell!)^2}\left\{\left[\frac{d^{\ell-1}}{dx^{\ell-1}}\left(x^2-1\right)^\ell\right]\left[\frac{d^\ell}{dx^\ell}\left(x^2-1\right)^\ell\right]\Bigg|_{-1}^{1}\right. \\
&\qquad\left. - \int_{-1}^{1} \left[\frac{d^{\ell-1}}{dx^{\ell-1}}\left(x^2-1\right)^\ell\right]\left[\frac{d^{\ell+1}}{dx^{\ell+1}}\left(x^2-1\right)^\ell\right] dx\right\} \\
&= \frac{1}{2^{2\ell}(\ell!)^2}\left\{\left[\frac{d^{\ell-2}}{dx^{\ell-2}}\ell\left(x^2-1\right)^{\ell-1}\cdot(2x)\right]\left[\frac{d^\ell}{dx^\ell}\left(x^2-1\right)^\ell\right]\Bigg|_{-1}^{1}\right. \\
&\qquad\left. - \int_{-1}^{1} \left[\frac{d^{\ell-1}}{dx^{\ell-1}}\left(x^2-1\right)^\ell\right]\left[\frac{d^{\ell+1}}{dx^{\ell+1}}\left(x^2-1\right)^\ell\right] dx\right\} \\
&= \frac{1}{2^{2\ell}(\ell!)^2}\left\{\frac{d^{\ell-3}}{dx^{\ell-3}}\left[\ell(\ell-1)\left(x^2-1\right)^{\ell-2}\cdot(2x)^2+\ell\left(x^2-1\right)^{\ell-1}\cdot(2)\right]\left[\frac{d^\ell}{dx^\ell}\left(x^2-1\right)^\ell\right]\Bigg|_{-1}^{1}\right. \\
&\qquad\left. - \int_{-1}^{1} \left[\frac{d^{\ell-1}}{dx^{\ell-1}}\left(x^2-1\right)^\ell\right]\left[\frac{d^{\ell+1}}{dx^{\ell+1}}\left(x^2-1\right)^\ell\right] dx\right\}.
\end{aligned}$$

Evaluating the rest of the derivatives in the first term above yields a sum of terms that each have at least one factor of x^2-1. Plugging in the boundary values, -1 and 1, results in zero.

$$\begin{aligned}
\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\,dx &= \frac{(-1)}{2^{2\ell}(\ell!)^2}\int_{-1}^{1} \left[\frac{d^{\ell-1}}{dx^{\ell-1}}\left(x^2-1\right)^\ell\right]\left[\frac{d^{\ell+1}}{dx^{\ell+1}}\left(x^2-1\right)^\ell\right] dx \\
&= \frac{(-1)}{2^{2\ell}(\ell!)^2}\left\{\overbrace{\left[\frac{d^{\ell-2}}{dx^{\ell-2}}\left(x^2-1\right)^\ell\right]}^{=0}\left[\frac{d^{\ell+1}}{dx^{\ell+1}}\left(x^2-1\right)^\ell\right]\Bigg|_{-1}^{1}\right. \\
&\qquad\left. - \int_{-1}^{1} \left[\frac{d^{\ell-2}}{dx^{\ell-2}}\left(x^2-1\right)^\ell\right]\left[\frac{d^{\ell+2}}{dx^{\ell+2}}\left(x^2-1\right)^\ell\right] dx\right\} \\
&= \frac{(-1)^2}{2^{2\ell}(\ell!)^2}\int_{-1}^{1} \left[\frac{d^{\ell-2}}{dx^{\ell-2}}\left(x^2-1\right)^\ell\right]\left[\frac{d^{\ell+2}}{dx^{\ell+2}}\left(x^2-1\right)^\ell\right] dx \\
&\ \ \vdots \\
&= \frac{(-1)^\ell}{2^{2\ell}(\ell!)^2}\int_{-1}^{1} \left[\frac{d^{\ell-\ell}}{dx^{\ell-\ell}}\left(x^2-1\right)^\ell\right]\left[\frac{d^{\ell+\ell}}{dx^{\ell+\ell}}\left(x^2-1\right)^\ell\right] dx
\end{aligned}$$

As a result,

$$\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\,dx = \frac{(-1)^\ell}{2^{2\ell}(\ell!)^2}\int_{-1}^{1}\left(x^2-1\right)^\ell\left[\frac{d^{2\ell}}{dx^{2\ell}}\left(x^2-1\right)^\ell\right]dx. \tag{1}$$

Take the 2ℓth derivative of $(x^2-1)^\ell$ for several values of ℓ and try to find a pattern.

$$\begin{aligned}
\ell = 0:&\quad \frac{d^0}{dx^0}(x^2-1)^0 = (x^2-1)^0 = 1\\
\ell = 1:&\quad \frac{d^2}{dx^2}(x^2-1)^1 = \frac{d^2}{dx^2}(x^2-1) = 2\\
\ell = 2:&\quad \frac{d^4}{dx^4}(x^2-1)^2 = \frac{d^4}{dx^4}(x^4-2x^2+1) = 24\\
\ell = 3:&\quad \frac{d^6}{dx^6}(x^2-1)^3 = \frac{d^6}{dx^6}(x^6-3x^4+3x^2-1) = 720\\
\ell = 4:&\quad \frac{d^8}{dx^8}(x^2-1)^4 = \frac{d^8}{dx^8}(x^8-4x^6+6x^4-4x^2+1) = 40\,320\\
&\quad \vdots\\
&\quad \frac{d^{2\ell}}{dx^{2\ell}}(x^2-1)^\ell = (2\ell)!
\end{aligned}$$

Then equation (1) becomes

$$\begin{aligned}
\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\,dx &= \frac{(-1)^\ell}{2^{2\ell}(\ell!)^2}\int_{-1}^{1}\left(x^2-1\right)^\ell (2\ell)!\,dx\\
&= \frac{(2\ell)!}{2^{2\ell}(\ell!)^2}\int_{-1}^{1}\left(1-x^2\right)^\ell dx\\
&= \frac{2(2\ell)!}{2^{2\ell}(\ell!)^2}\int_{0}^{1}\left(1-x^2\right)^\ell dx.
\end{aligned}$$

Make the following substitution.

$$\begin{aligned}
x &= \sin\theta \quad\rightarrow\quad 1-x^2 = 1-\sin^2\theta = \cos^2\theta\\
dx &= \cos\theta\,d\theta
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\,dx &= \frac{2(2\ell)!}{2^{2\ell}(\ell!)^2}\int_{\sin^{-1}(0)}^{\sin^{-1}(1)}\left(\cos^2\theta\right)^\ell\cos\theta\,d\theta\\
&= \frac{2(2\ell)!}{2^{2\ell}(\ell!)^2}\int_{0}^{\pi/2}\cos^{2\ell+1}\theta\,d\theta\\
&= \frac{2(2\ell)!}{2^{2\ell}(\ell!)^2}\left[\frac{(2\ell)!!}{(2\ell+1)!!}\right].
\end{aligned}$$

For $\ell = \ell'$, then,

$$\begin{aligned}
\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\,dx &= \frac{2(2\ell)!}{2^{2\ell}(\ell!)^2}\left[(2\ell)!!\cdot\frac{1}{(2\ell+1)!!}\right] \\
&= \frac{2(2\ell)!}{2^{2\ell}(\ell!)^2}\left[(2\ell)!!\cdot\frac{1}{(2\ell+1)(2\ell-1)(2\ell-3)\cdots(3)(1)}\right] \\
&= \frac{2(2\ell)!}{2^{2\ell}(\ell!)^2}\left[(2\ell)!!\cdot\frac{(2\ell)(2\ell-2)(2\ell-4)\cdots(4)(2)}{(2\ell+1)(2\ell)(2\ell-1)(2\ell-2)(2\ell-3)(2\ell-4)\cdots(4)(3)(2)(1)}\right] \\
&= \frac{2(2\ell)!}{2^{2\ell}(\ell!)^2}\left[\frac{(2\ell)^2(2\ell-2)^2(2\ell-4)^2\cdots(4)^2(2)^2}{(2\ell+1)(2\ell)!}\right] \\
&= \frac{2}{2\ell+1}\left[\frac{(2\ell)^2(2\ell-2)^2(2\ell-4)^2\cdots(4)^2(2)^2}{2^{2\ell}(\ell!)^2}\right] \\
&= \frac{2}{2\ell+1}\left[\frac{(2\ell)(2\ell-2)(2\ell-4)\cdots(4)(2)}{2^\ell \ell!}\right]^2 \\
&= \frac{2}{2\ell+1}\left\{\frac{[2(\ell)][2(\ell-1)][2(\ell-2)]\cdots[2(2)][2(1)]}{2^\ell \ell!}\right\}^2 \\
&= \frac{2}{2\ell+1}\left[\frac{2^\ell(\ell)(\ell-1)(\ell-2)\cdots(2)(1)}{2^\ell \ell!}\right]^2 \\
&= \frac{2}{2\ell+1}\left(\frac{2^\ell \ell!}{2^\ell \ell!}\right)^2 \\
&= \frac{2}{2\ell+1}.
\end{aligned}$$

Therefore,

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\,dx = \left(\frac{2}{2\ell+1}\right)\delta_{\ell\ell'}.$$